QUESTION 1 (9 Marks)

Marks

(a) In a set of 7 letters, some of the letters are T 's and all other letters are different. If the number of different arrangements of these letters is 210, how many letters are T 's. **2**

(b) In a colony of bacteria, the rate of change of the colony is given by:

$$\frac{dP}{dt} = kP - r,$$

where P is the number of bacteria at time t minutes, r is the constant rate per minute at which the bacteria die and k is a constant.

(i) Verify that $P = \frac{r}{k} - \frac{A}{k}e^{kt}$ is the solution to the rate equation $\frac{dP}{dt} = kP - r$, given A is a constant. **2**

(ii) Find the time when the population of the bacteria colony is reduced to zero, given that when $t = 0$, $P = 5000$, $k = 0\cdot 2$ and $r = 1500$. Give your answer to the nearest second. **3**

(iii) Find P when $t = 2$, (answer to the nearest bacteria). **2**

QUESTION 2 (9 Marks) START A NEW PAGE

Marks

(a) The velocity v cms^{-1} of a particle is given by $v = 2x + 5$. If the initial displacement is 1cm to the right of the origin, find the displacement as a function of time. **3**

(b) (i) A Brine solution contains 1kg of salt per 10 litres. It runs into a tank, initially filled with 500 litres of fresh water, at a rate of 25 litres per minute. At the same time, the mixture runs out of the tank at the same rate. If A kg is the amount of salt in the tank at time t minutes, Explain why: $\frac{dA}{dt} = 2.5 - \frac{A}{20}$. **2**

(ii) Find the amount of salt in the tank at the end of 60 minutes, assuming the mixture is kept homogenous (to the nearest 10 grams). **3**

(iii) Find the maximum concentration of salt in the mixture. **1**

QUESTION 3 (9 Marks) START A NEW PAGE

Marks

(a) Sixteen of the chickens on the James Ruse School Farm are separated at random into 4 pens of 4 chickens for a feed trial. **3**
What is the probability that 4 particular chickens, *A, B, C* and *D* are in 4 separate pens?

(b) The velocity of a body, v ms^{-1}, moving in a straight line is given as
$v = e^t - e^{-t}$, where t is the time in seconds.
The initial position of the body is at the origin.

(i) Find the displacement x as a function of time t. **2**

(ii) Find the acceleration when $t = 2$. **2**
Give your answer correct to 2 decimal places.

(iii) Show that the body does not have a zero acceleration. **2**

QUESTION 4 (9 Marks) START A NEW PAGE

Marks

The depth of water in y metres on a tidal creek is given by:

$$4\frac{d^2y}{dt^2} = 5 - y$$, where time t is measured in hours.

(i) Prove that the vertical motion of the water level is simple harmonic and hence find the centre of motion. **2**

(ii) Find the period of the motion. **1**

(iii) Given that $y = 2$ at low tide and $y = 8$ at high tide, and that $y = a + b\cos nt$ is the solution of the equation: $4\ddot{y} = 5 - y$, write down the values of a, b and n. **3**

(iv) If the low tide is at 10 am, what is the earliest time after low tide that a fishing boat requiring a depth of 4 metres of water can enter the creek? **3**

QUESTION 5 (9 Marks) START A NEW PAGE

Marks

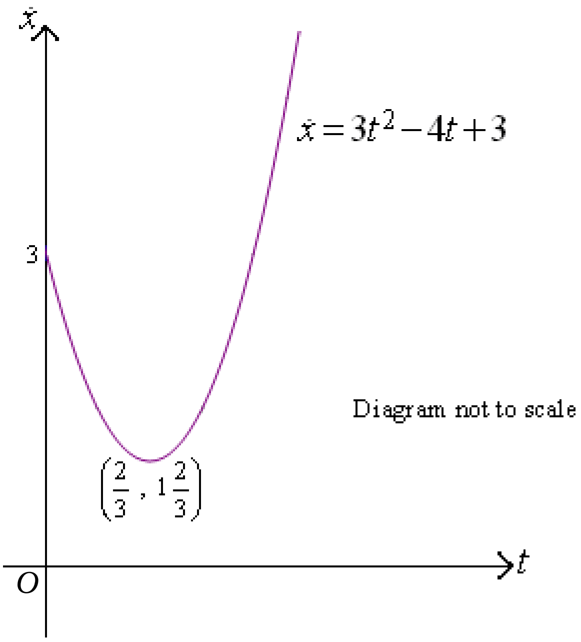
(a) Calculate the number of arrangements of the letters ***DESCARTES***:

(i) If the two ***S***'s are adjacent. **1**

(ii) If no two vowels are together. **2**

(iii) If the conditions from part (i) and (ii) hold simultaneously. **2**

(b) The graph below illustrates the velocity of a particle as a function of time.



(i) Sketch the graph of the particle to illustrate the acceleration as a function of time, given that the particle is initially 1 m to the left of the origin O. **2**

(ii) Hence write a description of the motion. **2**

QUESTION 6 (9 Marks) START A NEW PAGE

Marks

(a) The velocity v ms^{-1} of a particle moving along the x-axis is given by: $v = \sqrt{2 + 2\cos 2x}$. Initially the particle is located at the origin.

(i) Find the initial velocity and acceleration. **3**

(ii) Assuming that the particle reaches the position of $\frac{\pi}{2}$ metres from the origin, determine what would happen to the particle after this time. **2**

(b) In a certain experiment recording the number of bees N pollinating flowers in a given area, it was found that the rate of change of N is given by: $\frac{dN}{dt} = kN\left(1 - \frac{N}{2000}\right)$,
where t is the time in days and k is a constant.
At the beginning of the experiment 1000 bees were introduced to the area.

(i) Verify that $N = \frac{2000}{1+e^{-kt}}$ is the solution of the equation. **2**

(ii) If N =1500 when t = 10, determine the time in days, when N = 1800. **2**

QUESTION 7 (9 Marks) START A NEW PAGE

Marks

(a) A shell is detonated on level ground throwing fragments with a speed V ms^{-1} in all directions.
After a time T, a fragment hits the ground at a distance M from the shell.
You may assume these parametric equations of motion:

$$x = Vt\cos\alpha \text{ and } y = Vt\sin\alpha - \frac{1}{2}gt^2$$

(i) Show that: $g^2T^4 - 4V^2T^2 + 4M^2 = 0$. **2**

(ii) Hence find, to 2 decimal places, the shortest period of time during which a man, standing 20 metres from the place where the shell bursts, is in danger when V = 25. Take g = 10. **3**

(b) Twelve politicians are seated at a round table. A committee of five is to be chosen. If each politician, for one reason or another, dislikes their immediate neighbours and refuses to serve on a committee with them, in how many ways can a compatible group of five politicians be chosen? **4**

END OF EXAMINATION

Year 12 Term 2 2008 Extension I Question 1

		mark	comment
(a)	Let number of t's $= t$ $\frac{7!}{t!} = 210$ $t! = \frac{7!}{210}$ $t! = 4!$ $t = 4$ $\therefore$ there are 4 t's	2	1 for correct equation 1 for solution

		mark	comment
(b)(i)	$P = \frac{r}{k} - \frac{A}{k}e^{kt}$ $\frac{dP}{dt} = -Ae^{kt}$ but $kP = r - Ae^{kt}$ $-Ae^{kt} = kP - r$ $\therefore \frac{dP}{dt} = kP - r$	2	1 for differentiation 1 for substitution
(b)(ii)	$P = \frac{r}{k} - \frac{A}{k}e^{kt}$ when $t = 0, P = 5000, K = 0.2, r = 1500$ $5000 = \frac{1500}{0.2} - \frac{A}{0.2}e^{0}$ $A = 500$ $P = 7500 - 2500e^{0.2t}$ when $P = 0$ $0 = 7500 - 2500e^{0.2t}$ $e^{0.2t} = 3$ $0.2t = \ln 3$ $t = \frac{\ln 3}{0.2}$ $t = 5.493....$ time = 5 min 30 sec	3	1 for value of A 1 for $\frac{\ln 3}{0.2}$ 1 for approx. time
(b)(iii)	$P = \frac{r}{k} - \frac{A}{k}e^{kt}$ when $t = 2$ $P = 7500 - 2500e^{0.2\times 2}$ $P = 3770.438.....$ Population = 3770 to nearest bacteria	2	1 for substitution 1 for evaluation of population

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EXT 1 TERM 2 2008. **MATHEMATICS: Question** 2

Suggested Solutions	Marks	Marker's Comments
a) $v = \frac{dx}{dt} = 2x+5$ $\frac{dt}{dx} = \frac{1}{2x+5}$ $\underline{t = \frac{1}{2}\ln(2x+5) + c}$	1	
When $t = 0,\ x = 1$ $\therefore 0 = \frac{1}{2}\ln 7 + C \quad \therefore \underline{c = -\frac{1}{2}\ln 7}$	1	
$\therefore t = \frac{1}{2}\ln\left(\frac{2x+5}{7}\right)$ $e^{2t} = \frac{2x+5}{7} \Rightarrow x = \underline{\frac{7e^{2t}-5}{2}}$	1	Several people left final answer in form $x = \frac{e^{2t+\ln 7}-5}{2}$ (Half mark docked)
b) i) $\frac{dA}{dt}$ = Rate In − Rate Out (A is amount of salt in kgs) Rate In $= 1 \times \frac{25}{10} = 2.5$ kg/min Rate Out $= 25 \times \frac{A}{500} = \frac{A}{20}$ kg/min.	1	No second mark was given if $A/20$ appeared from nowhere.
$\therefore \underline{\frac{dA}{dt} = 2.5 - \frac{A}{20}}$	1	
ii) Solution of this equation is $\underline{A = B + Ce^{-t/20}}$ where B, C constants	1	
When $t = 0,\ A = 0 \quad \therefore B = -C$ $\therefore A = B(1 - e^{-t/20})$ But $\frac{dA}{dt} = \frac{Be^{-t/20}}{20} = 2.5 - \frac{B}{20} + \frac{Be^{-t/20}}{20}$ $\therefore B = 50$ $\therefore \underline{A = 50(1 - e^{-t/20})}$	1	If gms used and rounded wrong, last mark NOT given. If just a 3rd digit error in kgms only ½ mark deducted.
When $t = 60$, $A = 50(1 - e^{-3}) = \underline{47.51\ kg}$	1	
iii) As $t \to \infty$, $A \to 50$ as $e^{-t/20} \to 0$. $\therefore$ Max concentration, never strictly attained is $\frac{50}{500}$ kg/l $= \underline{0.1\ kg/l}$.	1	If stopped at A=50, then ½ given. Many people, incorrectly, used $\frac{dA}{dt} = 0$, and arrived at correct answer.

MATHEMATICS Extension 1 : Question 3

Suggested Solutions	Marks	Marker's Comments
a) (Pens identical) Place A, B, C, D in 4 pens (1 way). Choose groups of 3 to go in each pen from remaining 12 ways	3	① 369600
$= {}^{12}C_3 \times {}^{9}C_3 \times {}^{6}C_3 \times {}^{3}C_3 = 369600$		
No. of ways of choosing 4 groups of 4 from 16 (identical groups)		
$= \dfrac{{}^{16}C_4 \times {}^{12}C_4 \times {}^{8}C_4 \times {}^{4}C_4}{4!} = \dfrac{63063000}{4!}$		
$= 2627625$		① 2627625
$\therefore P(\text{ABCD separate}) = \dfrac{369600}{2627625} = \dfrac{64}{455}$		① 4!
Alternatively: pens not identical $\dfrac{4! \times 369600}{63063000} = \dfrac{64}{455}$		
Alternatively $\dfrac{16}{16} \times \dfrac{12}{15} \times \dfrac{8}{14} \times \dfrac{4}{13} = \dfrac{64}{455}$		
(A in a group) × (B in different group) × (C in diff gp) × (D in diff group)		
b) (i) $v = e^{t} - e^{-t}$	2	
$x = \int e^{t} - e^{-t}\, dt$		
$x = e^{t} + e^{-t} + c$		① $x = e^{t} + e^{-t} + c$
$x = 0 \quad t = 0 \quad \therefore c = -2$		① $c = -2$
$\therefore x = e^{t} + e^{-t} - 2$		
(ii) $a = \dfrac{dv}{dt} = e^{t} + e^{-t}$	2	① $a = e^{t} + e^{-t}$
$t = 2$, $a = 7.524391382$		① 7.52 (value, including 2dP)
accel is 7.52 m/s^2 (2dP)		no deduction if no units
(iii) $a = e^{t} + e^{-t}$		
$e^{t} > 0$ for all t	2	$e^{t} > 0$ ①
$e^{-t} > 0$ for all t		$e^{-t} > 0$ ①
$\therefore a > 0 \quad \therefore$ never zero		[insufficient to say $e^{t} \neq 0 \quad e^{-t} \neq 0$ ① only] or $e^{-t} \to 0 \quad e^{t} \to \infty$

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Graphs were accepted for ②.

QUESTION 4: (9 Marks)

(a) (i) $4\dfrac{d^2y}{dt^2} = 5 - y$

$\ddot{y} = -\frac{1}{4}(y-5)$ ①

Which is of the form $\ddot{y} = -n^2(x-b)$, hence the motion is SHM.

Centre of motion occurs when $\ddot{y} = 0$ i.e. at $y = 5$ ①

(ii) $n^2 = \frac{1}{4}$

$\therefore n = \frac{1}{2}$ (taking the positive value)

Period $= \dfrac{2\pi}{n} = 4\pi$ hours or 12hrs 34min. ①

(iii) $y = a + b\cos nt$

a is the centre of motion i.e. $a = 5$ ①

b is the amplitude $\frac{8-2}{2} = 3$

$n = \frac{1}{2}$ from (ii) ①

$\therefore b = -3$ negative sign measures time from low tide ①

$\therefore y = 5 - 3\cos\frac{1}{2}t$

$\therefore a = 5,\ b = -3,\ n = \frac{1}{2}$

(iv) Low tide occurs at 10am.

Fishing boat needs 4m of water i.e. $y = 4$

$\therefore 4 = 5 - 3\cos\frac{1}{2}t$ ①

$\frac{1}{3} = \cos\frac{1}{2}t$

$t \doteqdot$ 2hrs 28 minutes

$\therefore$ Required time from low tide is 2hrs 28 minutes and the actual time is 12:28pm. ①

Yr12 Ext 1 T2 (2008) Q5

5a) (i) 2 'e' as 1 unit

8 units with 2 'e' can be arranged in $\frac{8!}{2!} = 20160$ #

1m

(ii) 6 consonants with 2 's' can be arranged in $\frac{6!}{2!}$ ways

3 vowels A, E, E can occupy ${}^7C_3$ positions

Vowels can be ordered in $\frac{3!}{2!} = 3$ ways ($2!$ ← 2 'e')

No. of arrangements $= \frac{6!}{2!} \times {}^7C_3 \times 3 = 37800$ #

Alternatively ↑-↑-↑-↑-↑-↑-↑ 6 consonants 7 gaps

1st vowel has 7 positions

2nd 6 ..

3rd 5 ..

No. of arrangements $= \frac{6!}{2!} \times \frac{7 \times 6 \times 5}{2!} = 37800$ # ($2!$ ← 2 'e')

Any 2 of $\frac{6!}{2!} \times {}^7C_3 \times 2$ get 1m (with product evaluated)

$\frac{6!}{2!} \times 7 \times 6 \times 5 = 75600$ get 1m

2m

(iii) D C R T (SS) can be ordered in $5!$ ways

3 vowels can occupy ${}^6C_3$ positions

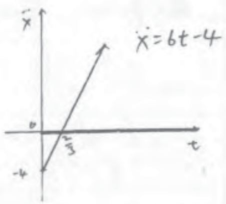
Vowels can be ordered in 3 ways

No of arrangements $= 5! \times {}^6C_3 \times 3 = 7200$

Alternatively $5! \times \frac{6 \times 5 \times 4}{2} = 7200$ (2 ← 2 'e')

$5! \times 6 \times 5 \times 4 = 14400$ get 1m

b i) 2m



No graph but show $\ddot{x} = 6t - 4$ $\frac{1}{2}$

$\frac{1}{2}$m for each intercept

straight line 1m

many students extend the straight line to left of $\ddot{x}$ axis lose $\frac{1}{2}$m ($t \geq 0$ only)

(ii) 2m Initially located 1m left of origin moving to the right ($v = 3$) $\frac{1}{2}$m

slowing down ($v > 0$ $a < 0$) $\frac{1}{2}$m

min velocity at $t = \frac{2}{3}$ s $\frac{1}{2}$m

Acceleration (force) change direction to be positive

Speeding up to the right ($v > 0$ $a > 0$) $\frac{1}{2}$m

bii) many students did not explain why particle move to right $V = 3 > 0$ slow down ($v > 0, a < 0$) speed up ($v > 0, a > 0$)

Complimentary Method

5 ii) 2 "e" eea eae aee

1 'e' 1 'a' ~~eae~~ eae eea aee

No of Restrictions − (2e + 1e 1a) + (eea + aee + eae)

$$\frac{9!}{2!\,2!} - \left[\underset{\uparrow\ 2e}{\frac{8!}{2!}} + \frac{8!}{2!} \times \underset{\uparrow\ a,e \text{ can swap}}{2}\right] + \left(\frac{7!}{2!} \times 3\right) \quad 1m$$

$= 90720 - [20160 + 40320] + 7560$

$= 37800$ 1m

MATHEMATICS Extension 1: Question 6

Suggested Solutions | **Marks** | **Marker's Comments**

6 (a) $v = \sqrt{2 + 2\cos 2x}$

(i) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2}(2 + 2\cos 2x)\right)$

$= \frac{d}{dx}(1 + \cos 2x)$

$= -2\sin 2x$ — 1

** For (a)(ii) see over page

OR $a = v \cdot \frac{dv}{dx}$

$= \sqrt{2+2\cos 2x} \cdot \frac{1}{2}(2 + 2\cos 2x)^{-\frac{1}{2}} \cdot -4\sin 2x$

$= -2\sin 2x$

When $t = 0$, $x = 0 \implies v = \sqrt{2 + 2\cos 0} = 2\text{ m/s}$ — 1

$\implies a = -2\sin 0 = 0\text{ m/s}^2$ — 1 — $\boxed{3}$

(b)(i) If $N = \frac{2000}{1+e^{-kt}}$ show $\frac{dN}{dt} = kN\left(1 - \frac{N}{2000}\right)$

Method 1:

$LHS = 2000(-1)(1+e^{-kt})^{-2} \cdot -ke^{-kt}$

$= \frac{2000ke^{-kt}}{(1+e^{-kt})^2}$ — 1

$RHS = kN\left(1 + \frac{N}{2000}\right)$

$= k \times \frac{2000}{1+e^{-kt}}\left(1 - \frac{1}{1+e^{-kt}}\right)$ where $\frac{N}{2000} = \frac{1}{1+e^{-kt}}$

$= \frac{2000k}{1+e^{-kt}}\left(\frac{1+e^{-kt} - 1}{1+e^{-kt}}\right)$ — 1

$= \frac{2000ke^{-kt}}{(1+e^{-kt})^2}$

$\therefore LHS = RHS$ as required — $\boxed{2}$

Method 2:

$LHS = \frac{2000ke^{-kt}}{(1+e^{-kt})^2}$

$= k \times \frac{2000}{1+e^{-kt}} \cdot \frac{e^{-kt}}{1+e^{-kt}}$

$= kN\left(\frac{1+e^{-kt} - 1}{1+e^{-kt}}\right)$

$= kN\left(1 - \frac{1}{1+e^{-kt}}\right)$ but $\frac{N}{2000} = \frac{1}{1+e^{-kt}}$

$= kN\left(1 - \frac{N}{2000}\right)$

$= RHS$ as required.

MATHEMATICS: Question 6 Extension 1

Suggested Solutions | **Marks** | **Marker's Comments**

6 Method 3:

$LHS = kN \frac{e^{-kt}}{1+e^{-kt}}$

$= kN\left(\frac{\frac{2000-N}{N}}{1 + \frac{2000-N}{N}}\right)$

$= kN\left(\frac{\frac{2000-N}{N}}{\frac{N+2000-N}{N}}\right)$

$= kN\left(\frac{2000-N}{2000}\right)$

$= kN\left(1 - \frac{N}{2000}\right)$

$= RHS$ as required

$N = \frac{2000}{1+e^{-kt}}$

$\frac{N}{2000} = \frac{1}{1+e^{-kt}}$

$\frac{2000}{N} = 1 + e^{-kt}$

$\frac{2000}{N} - 1 = e^{-kt}$

$\frac{2000-N}{N} = e^{-kt}$

b(ii) $N = 1500$, $t = 10 \implies 1500 = \frac{2000}{1+e^{-10k}}$

find k

$1 + e^{-10k} = \frac{4}{3}$

$e^{-10k} = \frac{1}{3}$

$-10k = \ln \frac{1}{3}$

$k = \frac{-1}{10}\ln\frac{1}{3}$

OR $k = \frac{1}{10}\ln 3$ — 1

When $N = 1800$, find t: $1800 = \frac{2000}{1+e^{-kt}}$ (where $k = \frac{1}{10}\ln 3$)

$1 + e^{-kt} = \frac{10}{9}$

$e^{-kt} = \frac{1}{9}$

$t = -\frac{1}{k}\ln \frac{1}{9}$

$\therefore t = \frac{1}{k}\ln 3^2$

$= \frac{2}{k}\ln 3$

$= \frac{2 \times 10}{\cancel{\ln 3}} \times \ln 3$

$t = 20$ hours

$t = \frac{5}{6}$ days — 1 — $\boxed{2}$ $-\frac{1}{2}$ if left in wrong units.

** (a)(ii) at $x = \frac{\pi}{2}$, $V = \sqrt{2 + 2\cos\pi} = 0$ — ½

$a = -2\sin\pi = 0$ — ½

$\therefore$ When particle reaches $x = \frac{\pi}{2}$ m, the particle will come to rest and remain there, as there is no force acting on the particle — 1 — $\boxed{2}$ Need to say why the particle remains at rest, otherwise $-\frac{1}{2}$ penalty.

QUESTION 7: (9 Marks)

(a) (i) Initial speed $V\ ms^{-1}$ at an angle of $\alpha°$

$x = vt\cos\alpha$

$t = T,\ x = m,\ y = 0$

$m = vT\cos\alpha$

$\cos\alpha = \dfrac{m}{VT}$

$\therefore \sin\alpha = \dfrac{\sqrt{v^2T^2 - m^2}}{VT}$ (1)

$y = vt\sin\alpha - \dfrac{gt^2}{2}$

$0 = VT\sin\alpha - \dfrac{gT^2}{2}$

$\therefore 0 = \cancel{VT} \cdot \dfrac{\sqrt{V^2T^2 - m^2}}{\cancel{VT}} - \dfrac{gT^2}{2}$

$\sqrt{V^2T^2 - m^2} = \dfrac{gT^2}{2}$

$V^2T^2 - m^2 = \dfrac{g^2T^4}{4}$

$\therefore g^2T^4 - 4V^2T^2 + 4m^2 = 0$ (1)

(ii) when $m = 20m$ $V = 25ms^{-1}$ $g = 10ms^{-2}$

$\therefore 100T^4 - 4\times(25)^2 + 4\times(20)^2 = 0$

$100T^4 - 2500T^2 + 1600 = 0$

$T^4 - 25T^2 + 16 = 0$ (1)

$T^2 = \dfrac{25 \pm \sqrt{25^2 - 4\times1\times16}}{2}$

$T^2 = \dfrac{25 \pm \sqrt{561}}{2}$

$T = 4.934,\ 0.811$ (1)

(1)

$\therefore$ Shortest period of time is 0.8secs. (nearest sec.)

(b) Case 1

If a particular politician, Mr X, is to be on the committee. The 2 people sitting next to Mr X will not be on the committee.

Therefore, there are 4 to choose from 9. (1)

Let C represent politicians who will be on the committee and C' those who will not be on the committee.

↑C'↑C'↑C'↑C'↑C'↑

(1) There are 6 places available for those on the committee i.e. ${}^6C_4$ ways of choosing the committee with Mr X on it.

Case 2:

(1)

If Mr X is not on the committee there are ${}^7C_5$ ways of choosing

$\therefore$ The total no. of ways $= {}^7C_5 + {}^6C_4$

$= 36$ (1)

7(b) (Method 2)

1 2 3 4 5 6 7 8 9

1358 7
1358
1359
1368
1369
1379
1468
1469
1479
1~~4~~5 9 9
2468
2469
2479
2579
3579

Ways to pick first person : 12

Ways to choose these 4 remaining

15 ways.

$\therefore$ total $= 15 \times 12$

$= 180$

But 1st person interchangeable with other 4 (which are fixed

So need to divide by

$\dfrac{5!}{4!}$

$\therefore$ Answer $= \dfrac{180}{\frac{5!}{4!}} = \underline{\underline{36.}}$